RADIATION REACTION IN CLASSICAL ELECTRODYNAMICS: THE CASE OF ROTATING CHARGED SPHERE

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In classical electrodynamics for rotating with variable angular velocity charged rigid sphere are found: the exact values of electromagnetic fields, the flux of radiating energy and the exact integral equation of rotation including the terms of self-rotation. The analysis of this eq. shows that on one hand, there is no "runaway" solutions, peculiar to the Lorentz-Dirac eq. of radiating point particle, on the other hand, there appear new problems, among them is the nonexistence of solution for some external moments of force.

1.

In 1998 there will be 60 years since the famous Dirac's work on relativistic radiation reaction force in classical electrodynamics had appeared. But the problems of relativistic radiation force are still discussed in the literature (see [1-10]). Among these problems are:

(i) can one consider the Lorentz-Dirac equation of motion for radiating charged point-like particle as the limit $R \to 0$ of usual equation of motion for extended charged body (R -is the size of the body)?

(- to our opinion [8] - the answer is negative: there is no analyticity near the point $R = 0$, thus one cannot mathematically strictly derive the Lorentz-Dirac equation)

(ii) while radiating, the system must lose its energy, so has the radiation reaction force the character of damping force or it can behavior in some cases as antidamping force? And if there is antidamping, do there exist runaway solutions of eq. of motion?

(some examples of antidamping one can find in [2,9,10])

To make a careful study of the problem of radiation reaction one must search for exact solutions of Maxwell equations for radiating charged systems. Here we consider one such exactly solvable case - the case of a charged sphere of radius R, rotating with variable angular velocity $\vec{\Omega} = \vec{\Omega}(t)$.

2.

Let the densities of charge and current be:

$$\rho = \frac{Q}{4\pi R^2}\delta(r-R)$$
$$\vec{j} = [\vec{\Omega}, \vec{r}]\rho \tag{1}$$

with $\vec{\Omega} = \Omega(t)\vec{e}_z$ - i.e. the rotation is the around the z-axis.

(In other words we fixed rigidly at initial moment of time the surface density of nonrotating rigid sphere to be constant and after this began the sphere to rotate.)

In Lorentz gauge the eq. for electromagnetic potentials ϕ and $\vec{A}$ are (latin indexes are $0,1,2,3$, greek -$1,2,3$, metric has the diagonal form $g_{ij} = diag(1,-1,-1,-1)$):

$$\partial_p\partial^p\varphi = 4\pi\rho$$
$$\partial_p\partial^p\vec{A} = 4\pi\vec{j}/c \tag{2}$$

Due to the form of current (1), the solution of (2) for $\vec{A}$ we can write as

$$\vec{A} = [\vec{e}_z, \vec{e}_r]B(t,r) \tag{3}$$

where the function B obeys the eq.:

$$\frac{\partial^2 B}{(c\partial t)^2} - \frac{\partial^2 B}{(\partial r)^2} - \frac{2\partial B}{r\partial r} + \frac{2B}{r^2} = \frac{Q\Omega(t)}{cR}\delta(r-R) \tag{4}$$

With the help of Fourier transformations:

$$B(t,r) = \int dw \exp{(-iwt)}B^*(w,r)$$

$$\Omega(t) = \int dw \exp{(-iwt)}\Omega^*(w)$$

equation (4) for $B^*(w,r)$ takes the form

$$\frac{\partial^2}{(\partial\xi)^2}B^* + \frac{2\partial}{\xi\partial\xi}B^* + (w^2 - \frac{2}{\xi^2})B^* = f \tag{5}$$

here $\xi = r/c$, $\xi_0 = R/c$ and $f = f(w,\xi) = \frac{Q}{c\xi_0}\Omega^*(w)\delta(\xi - \xi_0)$

The retarded solution of (5) is constructed with the help of Bessel and Hankel functions of semiwhole indexes:

$$B^* = \frac{\pi}{2i}\left[\frac{J_{3/2}(w\xi)}{\xi^{1/2}}\int_{\xi}^{\infty} dx x^{3/2} H^{(1)}_{3/2}(wx) + \frac{H^{(1)}_{3/2}(w\xi)}{\xi^{1/2}}\int_{0}^{\xi} dx x^{3/2} J_{3/2}(wx)\right] f(w,x) \tag{6}$$

The substitution of the inverse Fourier transformation for Ω^* and the use of the formula

$$\int dw \exp{(iwA)} J_{3/2}(wB) H^{(1)}_{3/2}(wC) =$$
$$i\frac{A^2 - B^2 - C^2}{2(BC)^{3/2}}\left(\epsilon(A+B+C) - \epsilon(A-B+C)\right) \tag{7}$$

with the step-function $\epsilon(x) = \left\{ \begin{smallmatrix} 1, x>0 \\ -1, x<0 \end{smallmatrix} \right\}$ (one can derive (7) using the standard representation of Bessel and Hankel functions of semiwhole indexes through the complex exponents)

gives at last for $B(t,r)$ the following result

$$B(t,r) = \left(-\frac{Qc^2}{4Rr^2}\right)\int_{t-|r+R|/c}^{t-|r-R|/c} dt'\Omega(t')\left[(t'-t)^2 - (r^2+R^2)/c^2\right] \tag{8}$$

The solution for φ (1,2) is obvious:

$$\varphi = \left\{ \begin{matrix} Q/r, & r > R \\ Q/R, & r < R \end{matrix} \right\} \tag{9}$$

Thus we have the exact solutions for the electromagnetic fields $\vec{E} = -\nabla\varphi - \frac{\partial\vec{A}}{c\partial t}$ and $\vec{H} = rot\ \vec{A}$:

$$\vec{E} = \vec{e}_r E_r + \vec{e}_\phi E_\phi, \quad \vec{H} = \vec{e}_r H_r + \vec{e}_\theta H_\theta$$

$$
E_r = \begin{Bmatrix} Q/r^2, & r > R \\ 0, & r < R \end{Bmatrix}, \quad E_\phi = -\sin\theta \frac{\partial B(t,r)}{c\partial t}
$$

$$
H_r = \frac{2\cos\theta B(t,r)}{r}, \quad H_\theta = \left(-\frac{\sin\theta}{r}\right)\frac{\partial(rB(t,r))}{\partial r} \tag{10}
$$

3.

The integration of the energy-momentum balance equation

$$
\partial_j T^{ij} = 0 \tag{11}
$$

over the space volume (here T^{ij} - the total (matter+ field) energy momentum tensor) gives the standard expression for the flux I of radiating energy through the sphere of radius r:

$$
I = \int r^2 d\Xi \frac{c}{4\pi}(\vec{e}_r[\vec{E},\vec{H}]) \tag{12}
$$

here $d\Xi = \sin\theta d\theta d\phi$ -the element of the space angle. The substitution of $\vec{E},\vec{H}$ from (10) into eq.(11) yields

$$
I = -\int r^2 \sin^3\theta d\theta d\phi \frac{1}{4\pi r}\frac{\partial B}{\partial t}\frac{\partial(rB)}{\partial r} \tag{13}
$$

In the wave zone ($r \to \infty$) one can rewrite function B (8) as

$$
B \approx \frac{QR}{2cr}\Omega(t - r/c) \tag{14}
$$

Consequently, with (14), the energy flux (13) is expressed as

$$
I = \frac{Q^2R^2}{6c^2}\left(\frac{\partial\Omega}{\partial t}\right)^2
$$

4.

The multiplication of the energy-momentum balance (11) with $i = \beta$ on $\epsilon_{\alpha\mu\beta}x^\mu$ and the integration over space volume V of the rotating sphere yields the equation of rotation in the form:

$$
\frac{d\vec{N}}{dt} = \vec{M} \tag{15}
$$

Here $\vec{M}$ - is the moment of external forces $\vec{M}_{ext}$ and of electromagnetic field $\vec{M}_{em}$:

$$\vec{M}_{em} = \frac{R^3}{4\pi} \int d\Xi \left([\vec{e}_r, \vec{E}](\vec{e}_r, \vec{E}) + [\vec{e}_r, \vec{H}](\vec{e}_r, \vec{H}) \right) \tag{16}$$

and $\vec{N}$ - is the angular momentum of the rotating rigid sphere $\vec{N}_{mech}$ with radius R and total mass M and of the internal electromagnetic field $\vec{N}_{em}$:

$$\vec{N}_{mech} = \int dV [\vec{r}, \vec{T}_{mech}]$$

$$\vec{T}_{mech} = \left(\frac{M}{4\pi r^2} \right) \frac{[\vec{\Omega}, \vec{r}] \cdot \delta(r - R)}{\sqrt{1 - (\Omega r \sin\theta / c)^2}} \tag{17}$$

$$\vec{N}_{em} = \int dV [\vec{r}, \vec{T}_{em}], \quad \vec{T}_{em} = \frac{[\vec{E}, \vec{H}]}{4\pi} \tag{18}$$

The integrand in (18) for $r \leq R$ (10) is proportional to $\vec{e}_\phi E_\phi H_r$, thus the integration over ϕ yields for it zero result - internal electromagnetic field gives zero contribution to the equation of rotation.

(Integrating (15) over the infinite sphere ($r \to \infty$) and taking into account (14), we get the rate of radiation of the total angular momentum of the system rigid sphere+field:

$$\frac{d\vec{N}}{dt} = -\frac{Q^2 R}{3c^2} \frac{\partial \vec{\Omega}}{\partial t}$$

)

Electromagnetic field (10)for $r = R$ with function B (8), which we can rewrite as

$$B(t, r = R) = \left(-\frac{Qc^2}{4R^3} \right) \int\limits_{t-R/c}^{t} dt' \Omega(t') \left[(t' - t)^2 - \frac{2R^2}{c^2} \right] =$$

$$\left(-\frac{Qc^2}{4R^3} \right) \int\limits_{0}^{2R/c} dx \Omega(t - x)(x^2 - \frac{2R^2}{c^2})$$

yields for $\vec{M}_{em}$ the result

$$\vec{M}_{em} = \vec{e}_z \left(\frac{Q^2 c}{6R^2} \right) \int\limits_{0}^{2R/c} dx \frac{\partial \Omega(t - x)}{\partial t} (x^2 - \frac{2R^2}{c^2}) \tag{19}$$

Consequently, after integration of (17), the equation of rotation (15) takes the form:

$$\frac{d}{dt}\left[\frac{MRc}{\Psi(t)}\left(\frac{\Psi^2(t)+1}{2\Psi(t)}\ln\frac{1+\Psi(t)}{1-\Psi(t)}-1\right)\right]=$$

$$\frac{d}{dt}\left[\left(\frac{Q^2c}{6R^2}\right)\int\limits_0^{2\eta}dx\Psi(t-x)(x^2-2\eta^2)\right]+M_{ext} \tag{20}$$

where $\Psi(t)=\Omega(t)R/c$ and $\eta=R/c$.

This is nonlinear integro-differential equation (intergal equation of self-interaction with retardation). Linearization of eq. (20) for $\Psi \ll 1$ yields:

$$\frac{d}{dt}\left[\frac{2MR^2}{3}\Omega(t)\right]=$$

$$\frac{d}{dt}\left[\left(\frac{Q^2c}{6R^2}\right)\int\limits_0^{2\eta}dx\Omega(t-x)(x^2-2\eta^2)\right]+M_{ext} \tag{21}$$

First terms in R.H.S. of (20-21) describe self-rotation of the sphere.

Thus we see that the self consistent treatment of radiation problem implies the non-point-like description of radiating system which leads inevitably to intergal equation of retardation (this fact was especially stressed in [11]).

5.

If $M_{ext}=0$ then eq.(20-21) are the eq. of self-rotation. Then in linear approximation eq. (21), rewritten as

$$\frac{d}{dt}\left[k\Omega(t)\right]=\frac{d}{dt}\left[\int\limits_0^{2\eta}dx\Omega(t-x)(x^2-2\eta^2)\right] \tag{22}$$

here $k=\frac{4MR^4}{Q^2c}$, has the solution

$$\frac{d}{dt}\left[\Omega(t)\right]=a\exp bt \tag{24}$$

with a, b- constants, a - arbitrary and b - is the negative ($b<0$) solution of the algebraic eq.

$$\frac{2MRc^2}{Q^2}=\frac{(\nu+1)^2}{\nu^3}\left(\frac{1-\nu}{1+\nu}-\exp\left(-2\nu\right)\right),\quad \nu=\frac{bR}{c}<-1$$

This solution describes the damping of the self-rotation (so there is no run-away solutions).

If $M_{ext} \neq 0$ then the complete solution of (21) is the sum of the (24) and of the partial solution of (21). Consequently, to fix arbitrary constant a, one must use some additional condition, which physical motivation, generally speaking, is not clear. Partial solution of (21) can be found by the Laplace transformation of (21), but as the theory of integral equations tells, not for every M_{ext} this partial solution exists! Thus we face new problem of radiation reaction.

6.

Following Lorentz-Dirac approach one can try to extract from eq. (21) the equation of rotation of point particle with radiation reaction. For this purpose let us introduce the angular momentum of point particle N as $N = N(t) = 2MR^2\Omega(t)/3$ and expand (12) in powers of R, $R \to 0$. So if eq.(21) takes the form $\frac{dN(t)}{dt}$ = (infinite with $R \to 0$ term, proportional to $\frac{dN(t)}{dt}$) + (finite with $R \to 0$ term, proportional to $\frac{d^2N(t)}{dt^2}$), then the first, infinite term leads to usual regularization, and the second, finite term is what one must take as radiation reaction.

With N eq. (21) is:

$$\frac{dN(t)}{dt} = \lambda \frac{d}{dt}\left[\int_0^{2\eta} dx N(t-x)(x^2 - 2\eta^2)\right] + M_{ext} \tag{25}$$

here $\lambda = \frac{Q^2 c}{4MR^4}$.

Expanding R.H.S. of (25) for $R \to 0$ (or $\eta \to 0$), we get:

$$\lambda \frac{d}{dt}\left[\int_0^{2\eta} dx N(t-x)(x^2 - 2\eta^2)\right] =$$

$$\lambda \frac{d}{dt}\left[\int_0^{2\eta} dx \left[N(t) - x\frac{dN(t)}{dt} + x^2/2\frac{d^2N(t)}{dt^2}\right](x^2 - 2\eta^2)\right] =$$

$$\frac{dN(t)}{dt}\left(-4\eta^3\lambda/3\right) + \frac{d^3N(t)}{dt^3}\left(8\eta^5\lambda/15\right)$$

From this expansion one can see that there is the infinite term, leading to regularization, but there are no finite with $R \to 0$ terms, so there is no radiation reaction, following Lorentz-Dirac approach.

Here we once more meet with the incompleteness of Lorentz-Dirac scheme (see also [8]).

7.

We conclude our treatment with the following remarks:

(i) self-consistent consideration of the problem of radiation reaction needs the non-point-like description of radiating system;

(ii) the non-point-like description does not coincide with Lorentz-Dirac scheme for infinitesimally small sizes of the body;

(iii) the non-point-like description of radiation reaction inevitably leads to integral equation of self-interaction;

(iv) integral equation of self-interaction possesses some peculiar features, thus solving some old problems of radiation reaction we can face new ones.

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